MATH 2060A TUT02

Mean Value Thm

Suppose
- f is cts on $[a,b]$
- f is diff. on (a,b)

Then $\exists\ c \in (a,b)$ s.t.

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Thm (6.2.7) Let $f: I \to \mathbb{R}$ be diff. on an interval I. Then

a) f is increasing on $I \iff f'(x) \geq 0\ \forall x \in I$

strictly $\uparrow$ $\;\not\Rightarrow\;\Leftarrow$ $f'(x) > 0\ \forall x \in I$

b) f is decreasing on $I \iff f'(x) \leq 0\ \forall x \in I$

strictly $\downarrow$ $\;\not\Rightarrow\;\Leftarrow$ $f'(x) < 0\ \forall x \in I$

Darboux's Thm

If
- f is diff. on $[a,b]$ and if
- k is a number between $f'(a)$ and $f'(b)$

then $\exists\ c \in (a,b)$ s.t. $f'(c) = k$

§ 6.2 15. Let I be an interval. Prove that if f is differentiable on I and if the derivative f' is bounded on I, then f satisfies a Lipschitz condition on I. (See Definition 5.4.4.)

Ans: Let $M > 0$ s.t. $|f'(x)| \leq M \quad \forall x \in I$

Let $x, y \in I$.

WLOG, assume $x < y$

Since f is cts on $[x, y]$ and diff. on (x, y),

by Mean Value Thm, $\exists\, c \in (x, y)$ s.t.

$$f(y) - f(x) = f'(c) \cdot (y - x).$$

Hence $$|f(y) - f(x)| = |f'(c)|\,|y - x| \leq M\,|y - x|.$$

Therefore f satisfies a Lipschitz condition on I. //

18. Let $I := [a,b]$ and let $f : I \to \mathbb{R}$ be differentiable at $c \in I$. Show that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x-y| < \delta$ and $a \le x \le c \le y \le b$, then

$$\left|\frac{f(x)-f(y)}{x-y} - f'(c)\right| < \varepsilon. \quad (*)$$

Ans: Note $\dfrac{f(x)-f(y)}{x-y} = \dfrac{f(x)-f(c)+f(c)-f(y)}{(x-c)+(c-y)}$

$$= \frac{x-c}{x-y}\cdot\frac{f(x)-f(c)}{x-c} + \frac{c-y}{x-y}\cdot\frac{f(c)-f(y)}{c-y}$$

and so

$$\frac{f(x)-f(y)}{x-y} - f'(c) = \frac{c-x}{y-x}\underbrace{\left(\frac{f(x)-f(c)}{x-c} - f'(c)\right)}_{\text{small}} + \frac{y-c}{y-x}\underbrace{\left(\frac{f(c)-f(y)}{c-y} - f'(c)\right)}_{\text{small.}}$$

Let $\varepsilon > 0$. Since f is diff. at c, $\exists\, \delta > 0$ s.t.
if $z \in [a,b]$ and $0 < |z-c| < \delta$,
then

$$\left|\frac{f(z)-f(c)}{z-c} - f'(c)\right| < \varepsilon$$

Now, if $0 < |x-y| < \delta$ and $a \le x < c < y \le \delta$

((*) is just def. of $f'(c)$ if one of x, y is c)

then $0 < |x-c| < \delta$ and $0 < |y-c| < \delta$ and so

$$\left|\frac{f(x)-f(y)}{x-y} - f'(c)\right| \le \frac{c-x}{y-x}\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| + \frac{y-c}{y-x}\left|\frac{f(c)-f(y)}{c-y} - f'(c)\right|$$

$$< \frac{c-x}{y-x}\varepsilon + \frac{y-c}{y-x}\varepsilon$$

$$= \varepsilon$$

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20. Suppose that $f : [0, 2] \to \mathbb{R}$ is continuous on $[0, 2]$ and differentiable on $(0, 2)$, and that $f(0) = 0$, $f(1) = 1$, $f(2) = 1$.

(a) Show that there exists $c_1 \in (0, 1)$ such that $f'(c_1) = 1$.

(b) Show that there exists $c_2 \in (1, 2)$ such that $f'(c_2) = 0$.

(c) Show that there exists $c \in (0, 2)$ such that $f'(c) = 1/3$.

Ans: a) Note f is cts on $[0,1]$ and diff. on $(0,1)$.

By MVT, $\exists\, c_1 \in (0,1)$ s.t.

$$f'(c_1) = \frac{f(1) - f(0)}{1 - 0} = 1$$

b) Note f is cts on $[1,2]$ and diff. on $(1,2)$.

By MVT, $\exists\, c_2 \in (1,2)$ s.t.

$$f'(c_2) = \frac{f(2) - f(1)}{2 - 1} = 0$$

c) Note $0 < c_1 < c_2 < 2$.

and f is diff. on $[c_1, c_2]$

Also, $f'(c_1) = 1 > \frac{1}{3} > 0 = f'(c_2)$

By Darboux Thm, $\exists\, c \in (c_1, c_2) \subseteq (0, 2)$

s.t. $f'(c) = \frac{1}{3}$

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§6.3

3. Let $f(x) := x^2 \sin(1/x)$ for $0 < x \le 1$ and $f(0) := 0$, and let $g(x) := x^2$ for $x \in [0, 1]$. Then both f and g are differentiable on [0,1] and $g(x) > 0$ for $x \neq 0$. Show that $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x)$ and that $\lim_{x \to 0} f(x)/g(x)$ does not exist.

Ans: By the same argument as in TUTO 1, it is easy to see that f is diff. on $[0,1]$. Also, g is clearly diff. on $[0,1]$. In particular, f, g are cts at 0 and hence

$$\lim_{x \to 0} f(x) = f(0) = 0, \quad \lim_{x \to 0} g(x) = g(0) = 0.$$

However, $\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ DNE by Sequential Criterion.

(Consider $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$.)

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5. Let $f(x) := x^2 \sin(1/x)$ for $x \neq 0$, let $f(0) := 0$, and let $g(x) := \sin x$ for $x \in \mathbb{R}$. Show that $\lim_{x \to 0} f(x)/g(x) = 0$ but that $\lim_{x \to 0} f'(x)/g'(x)$ does not exist.

Ans: $\forall x \neq 0$,

$$\left| \frac{f(x)}{g(x)} \right| = \left| \frac{x^2 \sin(\frac{1}{x})}{\sin x} \right| \leq |x| \cdot \left| \frac{x}{\sin x} \right| .$$

Note $\lim_{x \to 0} |x| \cdot \left| \frac{x}{\sin x} \right| = 0$ since $\lim_{x \to 0} \frac{x}{\sin x} = 1$.

By Squeeze Thm, $\lim_{x \to 0} \frac{f(x)}{g(x)} = 0$.

OTOH,
$$f'(x) = 2x \sin(\tfrac{1}{x}) - 2\cos(\tfrac{1}{x})$$
$$g'(x) = \cos x$$

If $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to 0} f'(x)$ also exists since $\lim_{x \to 0} g'(x) = 1$.

However $x_n := \frac{1}{n\pi} > 0$ $\forall n$ and $x_n \to 0$.

while $f'(x_n) = 0 - 2\cos(n\pi) = 2(-1)^{n+1}$

which is divergent. Contradiction!

Therefore $\lim_{x \to 0} \frac{f'(x)}{g'(x)}$ DNE //

We cannot "apply" L'Hôpital's rule blindly!

<u>Example</u> Prove that the eqn $1+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}=0$ has one real root if n is odd and no real root if n is even.

Ans: Let $g(x) := 1+x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{x^n}{n}$

Note g is cts and diff. on $\mathbb{R}$ with

$$g'(x) = 1+x+x^2+\cdots+x^{n-1} = \begin{cases} \frac{x^n-1}{x-1} & \text{if } x \neq 1 \\ n & \text{if } x = 1 \end{cases}$$

- Suppose n is odd,

  Then $g'(x) > 0 \quad \forall x \in \mathbb{R}$. ($x^n-1<0$ if $x<1$; $x^n-1>0$ if $x>1$)

  So g is strictly increasing on $\mathbb{R}$, and $g(x)=0$ has at most 1 real root.

  OTOH, since $\lim_{x\to-\infty} g(x) = -\infty$ (since n odd) and $\lim_{x\to\infty} g(x) = \infty$, Intermediate Value Thm implies that $g(x)=0$ has at least 1 real root.

  Hence $g(x)=0$ has exactly one real root

- Suppose n is even.

  Then $g'(-1) = 0$.

  Moreover, if $x<-1$, then $x^n-1>0$, $x-1<0 \Rightarrow g'(x)<0$

  if $-1<x<0$, then $x^n-1<0$, $x-1<0 \Rightarrow g'(x)>0$

  if $x \geq 0$, then $g'(x) \geq 1 > 0$.

  So g has global min. at $x=-1$.

  Now, $\forall x \in \mathbb{R}$,

$$\begin{aligned} g(x) \geq g(-1) &= 1+(-1)+\frac{(-1)^2}{2}+\frac{(-1)^3}{3}+\cdots+\frac{(-1)^n}{n} \\ &= (1-1)+\left(\frac{1}{2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n-2}-\frac{1}{n-1}\right)+\frac{1}{n} \quad (n \text{ even}) \\ &\geq \frac{1}{n} > 0. \end{aligned}$$

  Hence $g(x)=0$ has no real root. //